

# Symplectic aspects of polar actions

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## Abstract

An isometric compact group action  $G \times (M, g) \rightarrow (M, g)$  is called polar if there exists a closed embedded submanifold  $\Sigma \subseteq M$  which meets all the orbits orthogonally. Let  $\Pi$  be the associated generalized Weyl group. We study the properties of the lifting action  $G$  on the cotangent bundle  $T^*M$ . In particular, we show that the restriction map  $(C^\infty(T^*M))^G \rightarrow (C^\infty(T^*\Sigma))^\Pi$  is a surjective homomorphism of Poisson algebras. As a corollary, the singular symplectic reductions  $T^*M // G$  and  $T^*\Sigma // \Pi$  are isomorphic as stratified symplectic spaces, which gives a partial answer to a conjecture of Lerman, Montgomery and Sjamaar.

## 1 Introduction

Let  $(M, g)$  be a complete Riemannian manifold and  $G$  a compact Lie group acting on  $M$  by isometries. This action is called polar if there exists a closed embedded submanifold  $\Sigma \subseteq M$  meeting all orbits orthogonally ([7]). Then  $M$  is called a polar  $G$ -manifold and such a submanifold  $\Sigma$  is called a section and comes with a natural action by a discrete group of isometries  $\Pi = \Pi(\Sigma)$ , called its generalized Weyl group. Recall that by definition,  $\Pi(\Sigma) := N(\Sigma)/Z(\Sigma)$ , where

$$\begin{aligned} N(\Sigma) &= \{g \in G | g\Sigma = \Sigma\}, \\ Z(\Sigma) &= \{g \in G | gx = x, x \in \Sigma\}. \end{aligned}$$

Polar actions have nice properties and have been studied by many people, see for instance [2], [3], [6], [7], [8]. Basic examples of polar actions are the adjoint action of a compact Lie group on its Lie algebra. More generally, isotropy representations of symmetric spaces are also polar. It's a classical theorem of Dadok [2] which shows that a polar representation is (up to orbit equivalence) the isotropy representation of a symmetric space. An important feature of polar actions is the following Chevalley Restriction Theorem [7].

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**Theorem 1.1** *Let  $(M, g)$  be a polar  $G$ -manifold with a section  $\Sigma$  and generalized Weyl group  $\Pi$ . Then the following restriction to  $\Sigma$  is an isomorphism:*

$$|_{\Sigma} : C^{\infty}(M)^G \rightarrow C^{\infty}(\Sigma)^{\Pi},$$

where  $C^{\infty}(M)^G$  is the algebra of  $G$ -invariant smooth functions on  $M$ .

For a generalization of Chevalley Restriction Theorem to tensors, see [6].

In this paper we study symplectic aspects of polar actions. More precisely, we are looking at the lifting action of  $G$  on the cotangent bundle  $T^*M$  with its canonical symplectic structure  $\omega$ . This action is a Hamiltonian action with a moment map given by  $u : T^*M \rightarrow \mathfrak{g}^*$  with

$$u_X(x, \xi) = \langle \xi, X^*(x) \rangle, \quad (1.1)$$

where  $\mathfrak{g}$  is the Lie algebra of  $G$ ,  $\mathfrak{g}^*$  is the dual of  $\mathfrak{g}$  and  $u_X(x, \xi) = \langle u, X \rangle(x, \xi)$ ,  $X \in \mathfrak{g}$ . Moreover,  $X^*$  is the vector field on  $M$  generated by  $X$ . The moment map satisfies the following equations:

$$\begin{aligned} du_X &= i_{X^{\#}} \omega, \\ u(g \cdot (x, \xi)) &= Ad_g^* \cdot u(x, \xi), \quad \forall g \in G, \end{aligned}$$

where  $X^{\#}$  is the vector field on  $T^*M$  generated by  $X$ .

Our starting point is the following observation.

**Proposition 1.1** *Let  $(M, g)$  be a polar  $G$ -manifold with a section  $\Sigma$ . Then  $T^*\Sigma$  meets all  $G$ -orbits of the action  $G \times u^{-1}(0) \rightarrow u^{-1}(0)$ .*

Here  $T^*\Sigma$  is seen as a submanifold of  $T^*M$  under the natural isomorphism  $T^*M \cong TM$  induced by the Riemannian metric  $g$ . Note that in general  $T^*\Sigma$  can not meet all orbits of the lifting action of  $G$  on  $T^*M$  as it is easy to see that  $T^*\Sigma \subseteq u^{-1}(0)$  from (1.1).

Let  $C^{\infty}(T^*M)^G$  be the algebra of  $G$ -invariant smooth functions on  $T^*M$ . It carries a natural Poisson algebra structure with Poisson bracket  $\{f, g\} := \omega(X_f, X_g)$ , where  $X_f$  is the Hamiltonian vector field of  $f$  satisfying  $i_{X_f} \omega = df$ . The generalized Weyl group  $\Pi$  of the  $G$ -action on  $M$  also acts on  $T^*\Sigma$ . Let  $C^{\infty}(T^*\Sigma)^{\Pi}$  be the algebra of  $\Pi$ -invariant smooth functions on  $T^*\Sigma$  with its natural Poisson algebra structure. Our main result is the following symplectic analogue of Chevalley Restriction Theorem.

**Theorem 1.2** *Let  $(M, g)$  be a polar  $G$ -manifold with a section  $\Sigma$  and generalized Weyl group  $\Pi$ . Then the following restriction to  $T^*\Sigma$  is a surjective homomorphism of Poisson algebras:*

$$|_{T^*\Sigma} : C^{\infty}(T^*M)^G \rightarrow C^{\infty}(T^*\Sigma)^{\Pi}.$$

The above restriction to  $T^*\Sigma$  is not injective in general as  $T^*\Sigma \subseteq u^{-1}(0) \neq T^*M$  unless  $u \equiv 0$ .

The symplectic reduction  $T^*M // G := u^{-1}(0)/G$  is not a smooth manifold in general. However, it's a stratified symplectic space defined in [11]. The reader is referred to [11] for the precise definition of stratified symplectic spaces. A basic example is given by

$$X_0 = X // G := J^{-1}(0)/G,$$

where  $X$  is a Hamiltonian  $G$ -space with a moment map  $J : X \rightarrow \mathfrak{g}^*$ . Following [11], we define a function  $f_0 : X_0 \rightarrow \mathbb{R}$  to be smooth if there exists a function  $F \in C^\infty(X)^G$  with  $F|_{J^{-1}(0)} = \pi^* f_0$ , where  $\pi : J^{-1}(0) \rightarrow J^{-1}(0)/G$  is the projection map. In other words,  $C^\infty(X_0)$  is isomorphic to  $C^\infty(X)^G / I^G$ , where  $I^G$  is the ideal of  $G$ -invariant smooth functions on  $X$  vanishing on  $J^{-1}(0)$ . The algebra  $C^\infty(X_0)$  inherits a Poisson algebra structure from  $C^\infty(X)$ .

Let  $G$  and  $H$  be Lie groups and  $X$ , resp.  $Y$ , be smooth manifolds on which  $G$ , resp.  $H$  act properly. The stratified symplectic spaces  $T^*X // G$  and  $T^*Y // H$  are isomorphic if there exists a homeomorphism  $\phi : T^*X // G \rightarrow T^*Y // H$  and the pullback map

$$\begin{aligned} \phi^* : C^\infty(T^*Y // H) &\rightarrow C^\infty(T^*X // G), \\ f &\mapsto f \circ \phi \end{aligned}$$

is an isomorphism of Poisson algebras.

In [5] (Page 13, Conjecture 3.7), they made the following conjecture.

**Conjecture 1.1** *Let  $G$  and  $H$  be Lie groups and  $X$ , resp.  $Y$  be smooth manifolds on which  $G$ , resp.  $H$  act properly. Assume that the orbit spaces  $X/G$  and  $Y/H$  are diffeomorphic in the sense that there exists a homeomorphism  $\phi : X/G \rightarrow Y/H$  such that the pullback map  $\phi^*$  is an isomorphism from  $C^\infty(Y/H) := C^\infty(Y)^H$  to  $C^\infty(X/G) := C^\infty(X)^G$ . Then  $T^*X // G$  and  $T^*Y // H$  are isomorphic.*

Using Theorem 1.2, we can give a partial answer to conjecture 1.1. More precisely, we have the following corollary.

**Corollary 1.1** *Let  $M$  be a polar  $G$ -manifold with a section  $\Sigma$  and generalized Weyl group  $\Pi$ . Then  $T^*M // G$  and  $T^*\Sigma // \Pi$  are isomorphic.*

Under a slightly different assumption, it was proved that  $T^*M // G$  is homeomorphic to  $T^*\Sigma/\Pi$  in [5] (Proposition 3.8).

Proposition 1.1, Theorem 1.2 and Corollary 1.1 will be proved in section 3. A main ingredient of the proof is a characterization of symplectic slice representations of the lifting action  $G$  on  $T^*M$ , which is done by using the natural Sasaki metric on  $T^*M$ . Then combining the Multi-variable Chevalley restriction theorem proved by Tevelev [13] and other things, we are able to prove our results. For details, see section 3.

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## 2 Sasaki metrics on $TM$ and $T^*M$ .

In this section we describe the Sasaki metrics on  $TM$  and  $T^*M$  constructed in [9]. Given a Riemannian metric  $g$  on  $M$ , its Levi-Civita connection determines a splitting  $TTM = \mathcal{H}M \oplus \mathcal{V}M$ , where  $\mathcal{V}M = \ker d\pi$ ,  $\pi : TM \rightarrow M$  is the projection and  $\mathcal{H}M$  is spanned by  $X^h$ ,  $X$  is a smooth vector field on  $M$ . To describe  $X^h$ , let  $(x, v) \in TM$  and  $\gamma(t) : [0, 1] \rightarrow M$  be a smooth curve such that  $\gamma(0) = x$ ,  $\gamma'(0) = X(x)$ . Let  $Y(t) \in T_{\gamma(t)}M$  such that

$$\begin{cases} \nabla_{\gamma'} Y = 0 \\ Y(0) = v. \end{cases}$$

Then  $X^h(x, v) =: \bar{\gamma}'(0)$ , where  $\bar{\gamma}(t) = (\gamma(t), Y(t))$ . From the definition of  $X^h$ , we see that  $d\pi(X^h(x, v)) = X(x)$ . Let  $I_g$  be the natural isomorphism  $T^*M \cong TM$  induced by the Riemannian metric  $g$ . Then using the splitting  $TTM = \mathcal{H}M \oplus \mathcal{V}M \cong TM \oplus TM$ , we define the Sasaki metric  $\tilde{g}$  by

$$\tilde{g}(\langle X_1, X_2 \rangle, \langle Y_1, Y_2 \rangle) := g\langle X_1, Y_1 \rangle + g\langle X_2, Y_2 \rangle.$$

Define an almost complex structure  $J$  by setting  $J(X, Y) = (-Y, X)$ . Then  $\tilde{g}\langle J\cdot, J\cdot \rangle = \tilde{g}\langle \cdot, \cdot \rangle$  and the symplectic form  $\Omega := \tilde{g}(J\cdot, \cdot)$  is nothing but the pullback of  $\omega$  by the isomorphism  $I_g^{-1} : TM \cong T^*M$ , where  $\omega$  is the standard symplectic form on  $T^*M$ .

The Sasaki metric on  $T^*M$  is the pullback of  $\tilde{g}$  under the isomorphism  $I_g : T^*M \rightarrow TM$ . The following Lemma will be important for us.

**Lemma 2.1** *If  $\Sigma$  is a totally geodesic submanifold of  $(M, g)$ , then  $T\Sigma$  is a totally geodesic submanifold of  $(TM, \tilde{g})$ , where  $\tilde{g}$  is the Sasaki metric on  $TM$ .*

**Proof :** Let  $X$  be a smooth vector field on  $M$  such that  $X(x) \in T_x\Sigma$ ,  $\forall x \in \Sigma$ . As  $\Sigma$  is totally geodesic, we see that  $X^h|_{T\Sigma}$  is a smooth vector field on  $T\Sigma$  from the construction of  $X^h$ .

The vector field  $X$  also induces a vertical vector field  $X^\perp$  on  $TM$ . We choose a local coordinate to describe  $X^\perp$ . Let  $(x^1, \dots, x^n)$  be a local coordinate system at  $x \in M$ , where  $n = \dim M$ . Then any tangent vector  $v \in T_x M$  can be decomposed as  $v = v^i \frac{\partial}{\partial x_i}$ . The set of parameters  $\{x^1, \dots, x^n, v^1, \dots, v^n\}$  forms a natural coordinate system of  $TM$ . The natural frame in  $T_{(x, v)}TM$  is

given by  $\tilde{\partial}_i = \frac{\partial}{\partial x_i}$  and  $\tilde{\partial}_{n+i} = \frac{\partial}{\partial v_i}$ . Now if  $X = X^i \frac{\partial}{\partial x_i}$  is a vector field on  $M$ , then the vertical vector field  $X^\perp$  on  $TM$  is given by  $X^\perp = X^i \tilde{\partial}_{n+i}$ . As  $X(x) \in T_x \Sigma$ ,  $\forall x \in \Sigma$ , by definition we see that  $X^\perp|_{T\Sigma}$  is a vector field on  $T\Sigma$ .

To see that  $T\Sigma$  is totally geodesic in  $TM$ , choose two vector fields  $X, Y$  on  $M$  such that  $X(x), Y(x) \in T_x \Sigma$ ,  $\forall x \in \Sigma$ , then we have the following formula [4]:

$$\tilde{\nabla}_{X^\perp} Y^\perp = 0, \quad (2.2)$$

$$(\tilde{\nabla}_{X^h} Y^\perp)(x, v) = (\nabla_X Y)^\perp(x, v) + \frac{1}{2} R_x(v, Y_x, X_x)^h(x, v), \quad (2.3)$$

$$(\tilde{\nabla}_{X^\perp} Y^h)(x, v) = \frac{1}{2} (R_x(v, X_x, Y_x))^h(x, v), \quad (2.4)$$

$$(\tilde{\nabla}_{X^h} Y^h)(x, v) = (\nabla_X Y)^h(x, v) - \frac{1}{2} R_x(X_x, Y_x, v)^\perp(x, v), \quad (2.5)$$

where  $(x, v) \in T\Sigma$  and  $\nabla$ , resp.  $\tilde{\nabla}$  are Levi-Civita connections of  $g$ , resp.  $\tilde{g}$  and  $R$  is the Riemann curvature tensor of  $g$ .

Since  $\Sigma$  is totally geodesic, then  $\nabla_X Y(x), R_x(v, X_x, Y_x) \in T_x \Sigma$ . From (2.2) – (2.5), it follows that  $T\Sigma$  is totally geodesic.  $\square$

### 3 A symplectic analogue of Chevalley Restriction Theorem

In this section we prove Proposition 1.1, Theorem 1.2 and Corollary 1.1. A crucial property of polar actions we will use in the proof is the following result ([7] Theorem 4.6):

**Proposition 3.1** *Let  $M$  be a polar  $G$ -manifold with a section  $\Sigma$ . Then the slice representation at  $x$  is polar with a section  $T_x \Sigma$ ,  $\forall x \in \Sigma$ .*

Given Proposition 3.1, we can now give a proof of Proposition 1.1.

Recall that  $u : T^*M \rightarrow \mathfrak{g}^*$  is given by

$$u_X(x, \xi) = \langle \xi, X^*(x) \rangle.$$

Then for any  $(x, \xi) \in u^{-1}(0)$ ,  $\langle \xi, X^*(x) \rangle = 0$ ,  $\forall X \in \mathfrak{g}$ . Under the isomorphism  $I_g : T^*M \cong TM$  induced by the Riemannian metric  $g$ , the vector  $\xi^\# := I_g(\xi) \perp T_x(G \cdot x)$ , i.e.  $\xi^\# \in T_x(G \cdot x)^\perp$ .

As the isometric action  $G \times M \rightarrow M$  is polar with a section  $\Sigma$ , there exists  $h_1 \in G$  such that  $h_1 x \in \Sigma$ . Then  $h_1 \xi^\# \in T_{h_1 x}(G \cdot x)^\perp$ .

By Proposition 3.1, the slice representation

$$G_{h_1 x} \times T_{h_1 x}(G \cdot x)^\perp \rightarrow T_{h_1 x}(G \cdot x)^\perp$$

is polar. Hence there exists  $h_2 \in G_{h_1 x}$  such that  $h_2(h_1 \xi^\#) \in T_{h_1 x} \Sigma$ .

Let  $h = h_2 h_1$ , then

$$h(x, \xi^\#) = (hx, h\xi^\#) = (h_1 x, h_2 h_1 \xi^\#) \in T\Sigma.$$

So  $T^*\Sigma$  meets all orbits of the action  $G \times u^{-1}(0) \rightarrow u^{-1}(0)$ .

We proceed to give a proof of Theorem 1.2. Recall that we have a splitting  $TT^*M \cong \mathcal{H}M \oplus \mathcal{V}M$ , which induces an isomorphism  $TT^*M \xrightarrow{(d\pi, I_g)} TM \oplus TM$ , where  $d\pi$  is the differential of the projection  $T^*M \rightarrow M$  and  $I_g$  is the natural isomorphism  $T^*M \cong TM$  induced by the Riemannian metric  $g$ .

Let  $\{x^1, \dots, x^n, \xi_1, \dots, \xi_n\}$  be a local coordinate of  $T^*M$  at  $(x, \xi)$  and  $\Gamma_{ij}^k$  be the Christoffel symbols of the Levi-Civita connection  $\nabla$  induced by  $g$ . Then the horizontal lift of  $\frac{\partial}{\partial x_i}$  at  $(x, \xi)$  is given by

$$\tilde{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} + \Gamma_{il}^k \xi_k \frac{\partial}{\partial \xi^l}.$$

Here a horizontal lift of a vector  $X$  at  $(x, \xi)$  is defined to be the unique vector  $\tilde{X} \in \mathcal{H}M$  such that  $d\pi(x, \xi)(\tilde{X}) = X$ .

In terms of local coordinate system  $\{x^1, \dots, x^n, \xi_1, \dots, \xi_n\}$ , the almost complex structure  $J$  defined in section 2 can be rephrased as

$$\begin{aligned} J\left(\frac{\tilde{\partial}}{\partial x_i}\right) &= g_{ij} \frac{\partial}{\partial \xi^j}, \\ J\left(\frac{\partial}{\partial \xi^i}\right) &= -g^{ij} \frac{\tilde{\partial}}{\partial x_j}, \end{aligned}$$

where  $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$  and  $(g^{ij})$  is the inverse matrix of  $(g_{ij})$ .

Let  $X^* = X^i \frac{\partial}{\partial x_i}$  be a vector field on  $M$  generated by  $X \in \mathfrak{g}$ . Then the corresponding vector field on  $T^*M$  generated by  $X$  is

$$X^\#(x, \xi) = X^i \frac{\partial}{\partial x_i} - \sum_{i,j} \frac{\partial X^j}{\partial x_i} \xi_j \frac{\partial}{\partial \xi^i},$$

see [1] (Page 16 Lemma 11).

The Sasaki metric  $\tilde{g}$  on  $T^*M$  satisfies

$$\begin{aligned} \tilde{g}\left\langle \frac{\tilde{\partial}}{\partial x_i}, \frac{\tilde{\partial}}{\partial x_j} \right\rangle &= g_{ij}, \\ \tilde{g}\left\langle \frac{\tilde{\partial}}{\partial x_i}, \frac{\partial}{\partial \xi^j} \right\rangle &= 0, \\ \tilde{g}\left\langle \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right\rangle &= g^{ij}. \end{aligned}$$

**Lemma 3.1**  $\forall (x, \xi) \in T^*\Sigma$ , the Sasaki metric  $\tilde{g}$  on  $T^*M$  induces an orthogonal splitting

$$T_{(x, \xi)}T^*M = T_{(x, \xi)}(G \cdot (x, \xi)) \oplus JT_{(x, \xi)}(G \cdot (x, \xi)) \oplus V$$

with  $T_{(x, \xi)}T^*\Sigma \subseteq V$  and  $V$  is the orthogonal complement of  $T_{(x, \xi)}(G \cdot (x, \xi)) \oplus JT_{(x, \xi)}(G \cdot (x, \xi))$ .

**Proof:** Let  $X_i^\#$  be two vector fields on  $T^*M$  generated by  $X_i \in \mathfrak{g}$ ,  $i = 1, 2$  respectively and  $Y \in T_{(x, \xi)}T^*\Sigma$ . Then  $\tilde{g}\langle JX_1^\#, X_2^\# \rangle = \omega(X_1^\#, X_2^\#) = (i_{X_1^\#}\omega)(X_2^\#)$ . Let  $u$  be the moment map defined in (1.1), as  $(x, \xi) \in T^*\Sigma \subseteq u^{-1}(0)$ , by the  $G$ -equivalence of  $u$ , we get  $G(x, \xi) \subseteq u^{-1}(0)$ . Hence

$$\tilde{g}\langle JX_1^\#, X_2^\# \rangle = (i_{X_1^\#}\omega)(X_2^\#) = du_{X_1}(X_2^\#) = 0.$$

By the definition of  $J$ , we get  $JT_{(x, \xi)}T^*\Sigma \subseteq T_{(x, \xi)}T^*\Sigma$ . As  $T^*\Sigma \subseteq u^{-1}(0)$ , we get

$$\tilde{g}\langle X_1^\#, Y \rangle = \tilde{g}\langle JX_1^\#, JY \rangle = \omega(X_1^\#, JY) = (i_{X_1^\#}\omega)(JY) = du_{X_1}(JY) = 0.$$

Similarly,  $\tilde{g}\langle JX_1^\#, Y \rangle = 0$ . Hence  $T_{(x, \xi)}T^*\Sigma \subseteq V$ .  $\square$

The representation

$$G_{(x, \xi)} \times V \rightarrow V$$

is called the symplectic slice representation at  $(x, \xi)$ . Note that  $G_{(x, \xi)} = (G_x)_\xi =: \{h \in G_x \mid h\xi = \xi\}$ .

The following Lemma will be crucial for us.

**Lemma 3.2** Let  $M$  be a polar  $G$ -manifold with a section  $\Sigma$ . Then the symplectic slice representation at  $(x, \xi) \in T^*\Sigma$  is the diagonal action (up to identification)

$$(G_x)_{\xi^\#} \times (W \oplus W) \rightarrow W \oplus W,$$

where  $W := (G_x\xi^\#)^\perp$  is the orthogonal complement of  $G_x\xi^\#$  in the slice  $(G \cdot x)^\perp$ , i.e. we have

$$\begin{aligned} T_x M &= T_x(G \cdot x) \oplus (T_x(G \cdot x)^\perp), \\ T_x(G \cdot x)^\perp &= G_x\xi^\# \oplus (G_x\xi^\#)^\perp. \end{aligned}$$

**Proof :** Let  $G_{(x, \xi)} \times V \rightarrow V$  be the symplectic slice representation at  $(x, \xi)$ .

Under the isomorphism  $\Phi : \mathcal{H}M \oplus \mathcal{V}M \xrightarrow{(d\pi, I_g)} TM \oplus TM$ , we first claim that

$$\Phi(V) = W \oplus W.$$

Choose a local coordinate system  $\{x^1, \dots, x^n, \xi_1, \dots, \xi_n\}$  of  $T^*M$  at  $(x, \xi)$ . Then we have

$$\begin{aligned} d\pi\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_i}, \\ I_g\left(\frac{\partial}{\partial \xi^i}\right) &= g^{ij} \frac{\partial}{\partial x_j}. \end{aligned}$$

Let  $Z = a^i \frac{\tilde{\partial}}{\partial x_i} + b_i \frac{\partial}{\partial \xi^i} \in TT^*M$ . Then  $\Phi(Z) = (d\pi, I_g)(Z) = (a^i \frac{\partial}{\partial x_i}, g^{ij} b_i \frac{\partial}{\partial x_j}) =: (Y_1, Y_2)$ .

Let  $X^* = X^i \frac{\partial}{\partial x_i}$  be the vector field on  $M$  generated by  $X \in \mathfrak{g}$ , then the corresponding vector field on  $T^*M$  is

$$\begin{aligned} X^\#(x, \xi) &= X^i \frac{\partial}{\partial x_i} - \sum_{i,j} \frac{\partial X^j}{\partial x_i} \xi_j \frac{\partial}{\partial \xi^i} \\ &= X^i \frac{\tilde{\partial}}{\partial x_i} - X^i \Gamma_{il}^k \xi_k \frac{\partial}{\partial \xi^l} - \sum_{ij} \frac{\partial X^j}{\partial x_i} \xi_j \frac{\partial}{\partial \xi^i} \\ &= X^i \frac{\tilde{\partial}}{\partial x_i} - g \langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \frac{\partial}{\partial \xi^i}. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{g}(X^\#, Z) &= \tilde{g} \langle X^i \frac{\tilde{\partial}}{\partial x_i} - g \langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \frac{\partial}{\partial \xi^i}, a^j \frac{\tilde{\partial}}{\partial x_j} + b_j \frac{\partial}{\partial \xi^j} \rangle \\ &= g_{ij} X^i a^j - g^{ij} b_j g \langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \\ &= g \langle X^*, Y_1 \rangle - g \langle \nabla_{Y_2} X^*, \xi^\# \rangle. \end{aligned} \tag{3.6}$$

We also have

$$\begin{aligned} \tilde{g}(X^\#, JZ) &= \tilde{g} \langle X^i \frac{\tilde{\partial}}{\partial x_i} - g \langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \frac{\partial}{\partial \xi^i}, J(a^j \frac{\tilde{\partial}}{\partial x_j} + b_j \frac{\partial}{\partial \xi^j}) \rangle \\ &= \tilde{g} \langle X^i \frac{\tilde{\partial}}{\partial x_i} - g \langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \frac{\partial}{\partial \xi^i}, a^j g_{jk} \frac{\partial}{\partial \xi^k} - b_j g^{jk} \frac{\tilde{\partial}}{\partial x_k} \rangle \\ &= -X^i b_i - a^i g \langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \\ &= -g \langle X^*, Y_2 \rangle - g \langle \nabla_{Y_1} X^*, \xi^\# \rangle. \end{aligned} \tag{3.7}$$

Now we proceed to prove  $\Phi(V) = W \oplus W$ . Let  $Z \in TT^*M$  such that  $\Phi(Z) = (Y_1, Y_2) \in W \oplus W$ . We claim that  $Z \in V$  and it follows that  $W \oplus W \subseteq \Phi(V)$ . In fact, as  $(Y_1, Y_2) \in W \oplus W$ , we get

$$g \langle X^*, Y_1 \rangle = 0, \tag{3.8}$$

$$g \langle X^*, Y_2 \rangle = 0. \tag{3.9}$$

As  $X^*$  is a Killing vector field, we get

$$g \langle \nabla_{Y_2} X^*, \xi^\# \rangle = -g \langle \nabla_{\xi^\#} X^*, Y_2 \rangle = g \langle \nabla_{\xi^\#} Y_2, X^* \rangle, \tag{3.10}$$

and

$$g \langle \nabla_{Y_1} X^*, \xi^\# \rangle = g \langle \nabla_{\xi^\#} Y_1, X^* \rangle. \tag{3.11}$$



As  $M$  is a polar  $G$ -manifold with a section  $\Sigma$ , By Proposition 3.1, the slice representation  $G_x \times T_x(G \cdot x)^\perp \rightarrow T_x(G \cdot x)^\perp$  is polar with a section  $T_x\Sigma$ . Then by Proposition 3.1 again, the slice representation  $(G_x)_{\xi^\#} \times W \rightarrow W$  is polar with a section  $T_{\xi^\#}T_x\Sigma$ . As  $Y_1 \in W$ , there exists  $h \in (G_x)_{\xi^\#}$  such that  $hY_1 \in T_{\xi^\#}T_x\Sigma$ . Hence  $Y_1 \in h^{-1}(T_{\xi^\#}(T_x\Sigma)) = T_{\xi^\#}T_x(h^{-1}\Sigma) \cong T_x(h^{-1}\Sigma)$ . We also have  $\xi^\# = h^{-1}\xi^\# \in T_x(h^{-1}\Sigma)$ , as  $\Sigma$  is totally geodesic ([7], Theorem 3.2), so is  $h^{-1}\Sigma$ .

Then

$$g\langle \nabla_{\xi^\#} Y_1, X^* \rangle = g\langle B(\xi^\#, Y_1), X^* \rangle = 0, \quad (3.12)$$

$$g\langle \nabla_{\xi^\#} Y_2, X^* \rangle = g\langle B(\xi^\#, Y_2), X^* \rangle = 0 \quad (3.13)$$

where  $B(\cdot, \cdot)$  is the second fundamental form of  $h^{-1}\Sigma$ .

By (3.6), (3.8), (3.10) and (3.13), we get

$$\tilde{g}\langle X^\#, Z \rangle = g\langle X^*, Y_1 \rangle - g\langle \nabla_{Y_2} X^*, \xi^\# \rangle = 0.$$

Similary we get  $\tilde{g}\langle X^\#, JZ \rangle = 0$ . Hence  $\tilde{g}\langle JX^\#, Z \rangle = -g\langle X^\#, JZ \rangle = 0$ . It follows that  $Z \in V$ , which implies that  $W \oplus W \subseteq \Phi(V)$ .

On the other hand, we claim that  $\dim(W \oplus W) = \dim \Phi(V)$ . In fact,

$$\begin{aligned} \dim(W \oplus W) &= 2 \dim W \\ &= 2(\dim(T_x G \cdot x)^\perp - \dim(G_x \cdot \xi^\#)) \\ &= 2(\dim M - \dim G \cdot x - (\dim G_x - \dim(G_x)_{\xi^\#})) \\ &= 2 \dim M - 2(\dim G - \dim G_{(x, \xi)}), \end{aligned}$$

and

$$\begin{aligned} \dim \Phi(V) &= \dim V \\ &= \dim T^*M - 2 \dim G \cdot (x, \xi) \\ &= 2(\dim M - \dim G \cdot (x, \xi)) \\ &= 2(\dim M - (\dim G - \dim G_{(x, \xi)})). \end{aligned}$$

Hence  $\dim \Phi(V) = \dim(W \oplus W)$  and we have  $\Phi(V) = W \oplus W$ .

Now Lemma 3.2 follows from the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & W \oplus W \\ G_{(x, \xi)} \downarrow & & \downarrow (G_x)_{\xi^\#} \\ V & \xrightarrow{\Phi} & W \oplus W \end{array}$$

□

Given Lemma 3.2, we can now give a proof of Theorem 1.2. We first show that the following restriction map is surjective:

$$|_{T^*\Sigma} : C^\infty(T^*M)^G \rightarrow C^\infty(T^*\Sigma)^\Pi.$$

$\forall (x, \xi) \in T^*\Sigma$ , the Sasaki metric  $\tilde{g}$  on  $T^*M$  induces an orthogonal splitting

$$T_{(x, \xi)}T^*M = T_{(x, \xi)}G(x, \xi) \oplus JT_{(x, \xi)}G(x, \xi) \oplus V,$$

where  $\Phi(V) \cong W \oplus W$  by Lemma 3.2.

The Slice Theorem says that for an open  $G$ -invariant tubular neighborhood  $U_{(x, \xi)}$  of the orbit  $G(x, \xi)$ , there is a  $G$ -equivalent diffeomorphism

$$\exp^\perp : G \times_{G_{(x, \xi)}} S_{(x, \xi)}^\perp(\epsilon) \rightarrow U_{(x, \xi)},$$

where  $S_{(x, \xi)}^\perp := JT_{(x, \xi)}G(x, \xi) \oplus V$ ,  $S_{(x, \xi)}^\perp(\epsilon)$  is the  $\epsilon$ -ball in  $S_{(x, \xi)}^\perp$  and  $\exp^\perp$  is the normal exponential map of  $G(x, \xi)$ .

Let  $U = \bigcup_{(x, \xi) \in T^*\Sigma} U_{(x, \xi)}$ . As  $T^*\Sigma$  intersects all orbits in  $u^{-1}(0)$  by Proposition 1.1, we see that  $U$  is a  $G$ -invariant open neighborhood of  $u^{-1}(0)$ .  $\forall f \in C^\infty(T^*\Sigma)^\Pi$ , we first show that there exists  $F_\epsilon \in C^\infty(U_{(x, \xi)})^G$  such that

$$F_\epsilon|_{T^*\Sigma \cap U_{(x, \xi)}} = f|_{T^*\Sigma \cap U_{(x, \xi)}} \quad (3.14)$$

By the existence of  $G$ -invariant partition of unity subject to the cover  $U = \bigcup_{(x, \xi) \in T^*\Sigma} U_{(x, \xi)}$ , then there exists  $F \in C^\infty(U)^G$  such that  $F|_{T^*\Sigma} = f$ . Extending  $F$  to  $\tilde{F} \in C^\infty(T^*M)^G$ , we then prove our desired result.

To prove (3.14), we first recall some facts on polar representations which we will use. Let  $(G, K)$  be a symmetric pair and consider the isotropy representation of  $K$  on  $\mathfrak{p} = T_K(G/K)$ . It is a polar action and any maximal abelian sub-algebra  $\Sigma$  is a section. Its generalized Weyl group  $\Pi$  is also called the "baby" Weyl group. Consider the diagonal action of  $K$  on  $\mathfrak{p}^m$  (respectively  $\Pi$  on  $\Sigma^m$ ) and the corresponding algebra of invariant ( $m$ -variable) polynomials  $\mathbb{R}[\mathfrak{p}^m]^K$  (respectively  $\mathbb{R}[\Sigma^m]^\Pi$ ). Then we have the following result due to Tevelev [13].

**Theorem 3.1** *The restriction map  $|_\Sigma : \mathbb{R}[\mathfrak{p}^m]^K \rightarrow \mathbb{R}[\Sigma^m]^\Pi$  is surjective.*

As a polar representation is (up to orbit equivalence) the isotropy representation of a symmetric space [2]. Theorem 3.1 generalizes to the class of polar representations [6] (Corollary 2).

**Corollary 3.1** *Let  $K \subseteq O(\mathfrak{p})$  be a linear representation which is also polar with a section  $\Sigma$  and generalized Weyl group  $\Pi$ . Then the restriction is surjective:*

$$|_\Sigma : \mathbb{R}[\mathfrak{p}^m]^K \rightarrow \mathbb{R}[\Sigma^m]^\Pi.$$

**Corollary 3.2** *Let  $\mathfrak{p}$  be a polar representation of a compact Lie group  $K$  with a section  $\Sigma$  and generalized Weyl group  $\Pi$ . Then the restriction to  $\Sigma$  is surjective:*

$$|_{\Sigma} : C^{\infty}(\mathfrak{p}^m)^K \rightarrow C^{\infty}(\Sigma^m)^{\Pi}.$$

**Proof:** It's a classical result of Hillbert ([12], Proposition 2.4.14) that  $\mathbb{R}[\mathfrak{p}^m]^K$  is finitely generated. Let  $\rho_1, \dots, \rho_n$  be generators. By Corollary 3.1,  $\rho_1|_{\Sigma}, \dots, \rho_n|_{\Sigma}$  generate  $\mathbb{R}[\Sigma^m]^{\Pi}$ .

For any  $f \in C^{\infty}(\Sigma^m)^{\Pi}$ , apply Schwarz's Theorem [10] to the action of  $\Pi$  on  $\Sigma^m$ , we get  $F \in C^{\infty}(\mathbb{R}^n)$  such that  $f = F \circ \rho|_{\Sigma}$ , where  $\rho|_{\Sigma} : \Sigma^m \rightarrow \mathbb{R}^n$  be the map whose coordinates are  $\rho_1|_{\Sigma}, \dots, \rho_n|_{\Sigma}$ . Then  $\tilde{f} = F \circ \rho \in C^{\infty}(\mathfrak{p}^m)^K$  such that  $\tilde{f}|_{\Sigma} = f$ . □

We can now give a proof of (3.14). By Lemma 2.1, as  $\Sigma$  is totally geodesic in  $M$ , then  $T^*\Sigma$  is totally geodesic in  $T^*M$ . Hence the normal exponential map  $\exp^{\perp}$  of the orbit  $G \cdot (x, \xi)$  maps the  $\epsilon$ -ball  $B_{\epsilon}$  in  $T_{(x, \xi)}T^*\Sigma \cong T_x\Sigma \oplus T_x\Sigma$  diffeomorphically onto  $T^*\Sigma \cap U_{(x, \xi)}$ .  $\forall f \in C^{\infty}(T^*\Sigma)^{\Pi}$ ,  $f \circ \exp^{\perp} : B_{\epsilon} \rightarrow \mathbb{R}$  is a  $\Pi_{(x, \xi)}$ -invariant smooth function, where  $\Pi_{(x, \xi)} = \{h \in \Pi \mid h(x, \xi) = (x, \xi)\}$ . Let  $W$  be a polar representation of  $K := G_{(x, \xi)}$  with a section  $T_{\xi\#}T_x\Sigma \cong T_x\Sigma$  defined in Lemma 3.2. By Corollary 3.2, we see that there exists  $f_{\epsilon} \in C^{\infty}(W \oplus W)^K$  such that

$$f_{\epsilon}|_{B_{\epsilon}} = f \circ \exp^{\perp}$$

Hence  $f_{\epsilon} \circ (\exp^{\perp})^{-1} = f$  on  $T^*\Sigma \cap U_{(x, \xi)}$ . Combined with Lemma 3.2 and the Slice theorem, then  $f_{\epsilon}$  is pulled back to be a smooth function on  $G \times S_{(x, \xi)}^{\perp}(\epsilon)$  which descends to  $F_{\epsilon} \in C^{\infty}(U_{(x, \xi)})^G$  such that  $F_{\epsilon} = f$  on  $T^*\Sigma \cap U_{(x, \xi)}$ . We finish the proof of the surjectivity part of Theorem 1.2.

Let  $\omega$  be the standard symplectic form on  $T^*M$ . We show that the restriction to  $T^*\Sigma$  preserves Poisson brackets  $(C^{\infty}(T^*M)^G, \{, \}_1)$  and  $(C^{\infty}(T^*\Sigma)^{\Pi}, \{, \}_2)$ , where  $\{, \}_i$  are Poisson brackets induced by  $\omega$  and  $\omega|_{T^*\Sigma}$  respectively.

Let  $\mathring{M} \subseteq M$  be the union of principal orbits and  $\mathring{\Sigma} = \Sigma \cap \mathring{M}$ . Then  $\mathring{\Sigma}$  is open and dense in  $\Sigma$  ([3] Proposition 1.3). It follows that  $T^*\mathring{\Sigma} \subseteq T^*\Sigma$  is also open and dense.  $\forall (x, \xi) \in T^*\mathring{\Sigma}$ , we have the following orthogonal splitting with respect to the Sasaki metric  $\tilde{g}$  on  $T^*M$ :

$$T_{(x, \xi)}T^*M \cong T_{(x, \xi)}G(x, \xi) \oplus JT_{(x, \xi)}G(x, \xi) \oplus T_{(x, \xi)}T^*\mathring{\Sigma}. \quad (3.15)$$

To see this, as  $\mathring{\Sigma}$  consists of principal orbits, the slice representation at  $x \in \mathring{\Sigma}$  is trivial. Hence  $G_{(x, \xi)} = G_x$ ,  $\forall (x, \xi) \in T^*\mathring{\Sigma}$ . By [7], we also have  $\dim G \cdot x + \dim \mathring{\Sigma} = \dim M$ . Then the dimension of the vector space on the right hand side of (3.15) is equal to

$$2 \dim G(x, \xi) + 2 \dim \mathring{\Sigma} = 2(\dim G \cdot x + \dim \mathring{\Sigma}) = 2 \dim M$$

which finishes the proof of (3.15).

$\forall f \in C^\infty(M)^G$ , at  $(x, \xi) \in T^*\hat{\Sigma}$ , we can write

$$X_f = X + JY + Z,$$

where  $X, Y \in T_{(x, \xi)}G(x, \xi)$ ,  $Z \in T_{(x, \xi)}T^*\hat{\Sigma}$ .

Recall that  $i_{X_f}\omega = df$ ,  $\omega$  is the standard symplectic form on  $T^*M$ . Since  $f$  is  $G$ -invariant, we get  $(i_{X_f}\omega)(Y) = df(Y) = 0$ . Then

$$\omega(X_f, Y) = \tilde{g}(JX_f, Y) = \tilde{g}(JX - Y + JZ, Y) = -\tilde{g}(Y, Y).$$

It follows that  $Y = 0$ .

Now let  $f_1, f_2 \in C^\infty(T^*M)^G$ , at  $(x, \xi) \in T^*\hat{\Sigma}$ , we have

$$X_{f_i} = X_i + Z_i, \quad i = 1, 2$$

where  $X_i \in T_{(x, \xi)}G(x, \xi)$ ,  $Z_i \in T_{(x, \xi)}T^*\hat{\Sigma}$ .

Let  $\hat{f}_1 = f_1|_{T^*\hat{\Sigma}}$ , then we claim that  $X_{\hat{f}_1} = Z_1$ . In fact  $i_{X_{\hat{f}_1}}\omega|_{T^*\Sigma} = d\hat{f}_1$ .  $\forall Y \in T_{(x, \xi)}T^*\hat{\Sigma}$ , we have

$$\begin{aligned} \tilde{g}\langle X_{\hat{f}_1}, Y \rangle &= \omega(X_{\hat{f}_1}, JY) \\ &= d\hat{f}_1(JY) \\ &= df_1(JY) \\ &= \omega(X_{f_1}, JY) \\ &= \tilde{g}\langle Z_1, Y \rangle. \end{aligned}$$

Then at  $(x, \xi)$ ,

$$\begin{aligned} \{f_1, f_2\}_1 &= \omega(X_{f_1}, X_{f_2}) \\ &= \tilde{g}(JX_1 + JZ_1, X_2 + Z_2) \\ &= \tilde{g}(JZ_1, Z_2) \\ &= \omega|_{T^*\Sigma}(Z_1, Z_2) \\ &= \{\hat{f}_1, \hat{f}_2\}_2. \end{aligned}$$

By continuity,  $\{f_1, f_2\}_1 = \{\hat{f}_1, \hat{f}_2\}_2$  on  $T^*\Sigma$  everywhere.

**Proof of Corollary 1.1:** As  $M$  is a polar  $G$ -manifold, the inclusion:  $\Sigma/\Pi \rightarrow M/G$  is a homomorphism [7]. By Chevalley restriction theorem [7], the restriction  $|_\Sigma : C^\infty(M)^G \rightarrow C^\infty(\Sigma)^\Pi$  is an isomorphism, which implies that  $M/G$  is diffeomorphic to  $\Sigma/\Pi$ . We now show that  $T^*M // G$  is diffeomorphic to  $T^*\Sigma // \Pi$ . By Theorem 1.2,  $C^\infty(T^*M)^G/I^G$  is isomorphic to  $C^\infty(T^*\Sigma)^\Pi$  as Poisson algebra, where  $I^G$  is the ideal of  $G$ -invariant smooth functions on  $X$  vanishing on  $u^{-1}(0)$ .

It's enough to show that the inclusion  $T^*\Sigma // \Pi \rightarrow T^*M // G$  is a homomorphism. By Proposition 1.1, it suffices to show

$$G \cdot (x, \xi) \cap T^*\Sigma = \Pi \cdot (x, \xi), \quad \forall (x, \xi) \in T^*\Sigma.$$

Clearly  $\Pi \cdot (x, \xi) \subseteq G \cdot (x, \xi) \cap T^*\Sigma$ . On the other hand,  $\forall h_1(x, \xi) \in G(x, \xi) \cap T^*\Sigma$ , we have  $h_1x \in G \cdot x \cap \Sigma$  and  $h_1\xi^\# \in T_{h_1x}\Sigma$ . By Corollary 4.9 in [7], we get

$$G \cdot x \cap \Sigma = \Pi \cdot x, \quad \forall x \in \Sigma.$$

Hence

$$h_1x = h_2x, \quad h_2 \in \Pi. \quad (3.16)$$

Then  $(h_2^{-1}h_1)x = x$  and so  $h_2^{-1}h_1 \in G_x$ . Since  $(x, \xi) \in T^*\Sigma$ , we get  $\xi^\# \in T_x(G \cdot x)^\perp$ . By Lemma 3.2, the slice representation:  $G_x \times T_x(G \cdot x)^\perp \rightarrow T_x(G \cdot x)^\perp$  is polar with a section  $T_x\Sigma$  and generalized Weyl group  $\Pi_x$ . By Corollary 4.9 in [7] again,

$$G_x \cdot \xi^\# \cap T_x\Sigma = \Pi_x \cdot \xi^\#.$$

As  $h_2^{-1}h_1 \in G_x$ ,  $h_2 \in \Pi$ ,  $h_1\xi^\# \in T_{h_1x}\Sigma$ , we get  $h_2^{-1}h_1\xi^\# \in G_x \cdot \xi^\# \cap T_x\Sigma$ . Then there exists  $h_3 \in \Pi_x$  such that

$$h_2^{-1}h_1\xi^\# = h_3\xi^\#.$$

Hence  $h_1\xi^\# = h_2h_3\xi^\# \in \Pi \cdot \xi^\#$ . Combined with 3.16, we obtain  $h_1(x, \xi^\#) = (h_2x, h_2h_3\xi^\#) = h_2h_3(x, \xi^\#) \in \Pi \cdot (x, \xi^\#)$ . So  $G(x, \xi) \cap T^*\Sigma \subseteq \Pi \cdot (x, \xi)$ .  $\square$

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